An Analysis of Equity Portfolio Optimization with Correlation Matrices

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Abstract

Correlation is used frequently both in the classroom and in professional environments to illustrate and summarize investment know-how, especially with regard to diversification. Pedagogically, the initial build-up on correlation, which reaches its climax while describing a hypothetical two-variable optimization case, abruptly disappears when the discussion reaches optimizations of several securities, thereby stopping short of running a full-fledged, correlation-based optimization. Why is that so? We offer some explanations. First, correlations initially seem to provide clarification of the workings of the optimization, specifically with respect to how security risk-relations affect optimal weights. However, the variable transformation required changes coordinates, thus making correlation-based optimal weights and the desired information hard to understand. Second, correlation-based optimizations may be counterproductive. Nobody with a minimum of financial sophistication would try to make up covariance estimates; correlations, however, are easy to make up, which may make one overstate their practical value. Third, while mean-variance optimal weights can be easily constructed from correlation-based optimal numbers, not transforming the optimal numbers back to the mean-variance values deforms the information processed. We do not expect correlation-based optimizations to replace mean-variance ones except in specialized cases (e.g., small portfolios where investors may have extra-knowledge of security relationships).

Keywords: Portfolio optimization, mean-variance, correlations, regressions.

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1. Introduction

Diversification is the elusive reward that portfolio theory offers to investors. To study such potential diversification one can go directly to its source: the variance-covariance matrix quantifying relationships among securities. In the case of stocks and many other investments, this matrix is hard to understand because of the units in which the magnitudes are expressed (roughly corresponding to squared returns). A more attractive prospect is to look at the matrix of correlation coefficients, which is easy to understand and, perhaps because of that, popular enough to occasionally even appear in newspapers, e.g., [1]. A correlation coefficient quantifies how the two variables move together and is usually expressed in percentage form. Presumably, optimizing a portfolio using a correlation matrix would link optimal weights to correlations and enhance our understanding of the whole process. Unfortunately, such optimizations are nowhere to be seen (textbooks, literature), a rather peculiar absence that motivated our investigation. We provide correlation-based portfolio optimizations and study their practical and pedagogical value.

The correlation coefficient indicates how one variable moves with another. It is computed to range between +1 and -1. These extremes indicate full synchronization: positive, when the variables move up/down together; negative, when one variable moves up and the other moves down. Moving away from ±1 indicates same ―qualitative‖ movement but with less strength. The intermediate point, a correlation of zero, indicates that, based on the data being used, the movements of each variable do not seem to be co-related to the movements of the other variable. A correlation of zero is associated with ―insurance‖ --if a life insurance company selling policies to two individuals; it is best for them not to be related at all. It is appropriate to note that the concept ―relationship‖ is wider and more complex than that of co-relation, which is restricted to a paired, observation-by-observation association of numerical variables. The study of co-relationships using leads and lags is one of the more complex areas of econometric analysis.

The co-movement described in the previous paragraph applies to investments as well. Clearly, if one investment goes down, it would be good if the other investment doesn't follow. But if one investment goes down, a negative correlation would offer support to the idea that at least one of the assets will go up –this is the core of the ―diversification‖ concept. A third major risk management principle, ―hedging‖, is also best explained with the help of correlation. Establishing positions on perfectly negatively correlated assets –the perfect hedge, as in ―hedging your bets‖-- would offer the highest likelihood to having some up position at the end of the trading horizon. Derivative securities were custom-made for hedging: instead of buying and selling (or selling short) the same asset to get the perfectly negative correlation, one would take a given position in a given asset and the contrary position on its derivative (forward, futures, or options). Further, to cap it all, the correlation coefficient can easily be expressed as a percentage, which facilitates taking advantage of the information it conveys.

Again, why do we not run portfolio optimizations with correlation matrices? In order to answer, we must first examine the effects of certain transformations of variables. Next, we must evaluate the benefits and the limitations of using correlation matrices in portfolio optimization. What we discover in this analysis may not appear favorable to correlation-based optimizations. However, this does not imply correlation analysis has no usefulness. For example, correlation-based analysis may be helpful when considering portfolios of a few securities (limited diversification) where investors may have some extra-knowledge of variable relationships.

2. Variable Transformations, Regression Analysis, and Portfolio Optimization

Optimal portfolio weights can be obtained by maximizing the following unrestricted function:

$$F(x) = - \frac{1}{2} x' A x + x' b \quad (1)$$

This is a quadratic equation composed of a quadratic form (x' A x) and a vector (x' b), where A and b represent the covariance matrix and the vector of average individual stock returns, respectively. The first order conditions provide a set of simultaneous equations, A x = b. Optimal weights are calculated by normalizing the x auxiliary variables; that is, $w_i^* = x_i / \Sigma x_i^*$. These are the expressions for the portfolio return and its variance, respectively: $r_p = w' b$, and $var_p = w' A w$.

The optimal portfolio thus calculated is the so-called ―tangent‖ portfolio, which is the one that maximizes the return-to-risk ratio. This is also the portfolio that includes the usual (linear) arbitrage relationships ($\Sigma w_i = 1 = w_p$, $\Sigma w_i r_i = w_p r_p = r_p$), which implies that the portfolio cannot have more value than any of its parts. The algebraic formulation of portfolio optimization in (1) above keeps the analysis in the well-known area of simultaneous equations systems (SES), which is also shared by regression analysis.

We could think of applying linear algebra techniques to the first order conditions of the mean-variance model to study what types of equivalent transformations would change the usual mean-variance optimization into a standardized mean-correlation specification (C x = d; where C and d would now represent correlation matrix, and the vector of standardized means, respectively). As per common usage in linear algebra, equivalent transformations are those that do not alter the set of optimal solutions in a system of simultaneous equations. We started to pursue this course of action and, right away, we noticed that the potential changes would not only change the coordinate system of reference, but would alter the right-hand side by changing the units of the average returns as well. Unfortunately, equivalent transformations seem to cloud the optimization with seemingly arbitrary changes, and they cannot produce a straightforward way to re-state mean-variance results. A more advantageous tactic is to exploit the relationship between regression analysis and portfolio optimization to study the effects of

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certain variable transformations which, as it happens, have been well-known to statisticians since the dawn of econometric analysis. The regression approach to portfolio optimization was first developed by Jobson and Korkie [2], and further studied by Britten-Jones [3] and Tarrazo [4].

Tables 1 and 2 study the effects of some well-known transformations in the context of ordinary, multivariate regression (y = X b + u). Some results could be obtained using probability distributions and mathematical statistics but it is more advantageous to keep to straightforward linear algebra. Table 1 shows the matrix approach to ordinary least squares for both the original and the mean-adjusted variables. Note that mean-adjusting the regressors, but not the regressand, would produce the same slope estimates but higher fitting errors, which means we need to adjust means of all variables, or none. When we do so, the two regressions in Table 1 are exactly equivalent (same R-squared and associated goodness of fit indicators). Note the role of the interplay between the intercept, calculated with the usual vector of ones, and alternative mean specifications. We are using population‖ formulas for clarity. Table 2 is more interesting for our purposes. The top shows the multivariate regression for both mean-adjusted and standardized variables, which is often used to obtain variables thought to be normally-distributed, and with expected value and variance of 0 and 1, respectively. This regression is helpful because it shows very clearly how the transformed regression coefficients (bx_i*) are related to the original coefficients (bx_i). For reasons that will become clear later on, we would like to retain the mean vector in the optimization; therefore, the bottom regression is the one of highest interest at this point in the analysis.

3.Portfolio Optimizations Using Correlations: How

Tables 3 and 4 carry the analysis over to the portfolio optimization arena. The top of Table 3 shows the atypical regression that yields optimal portfolio weights. The data matrix X, which includes vectors x1, x2, and x3, represents security returns. We regress these returns on the y variable, which is simply a vector of ones (actually any constant would do) and calculate OLS estimates (bx). Then, we compute optimal portfolio weights (w* = [w_1* w_2* w_3*]') by normalizing betas: w_i* = bx_i / Σ bx_i*. The conventional mean-variance optimization appears at the bottom of Table 3 as well. It boils down to solving a simultaneous equation system (Ax = b) which, through a normalization, produces the optimal portfolio weights.

Table 1: Initial data and deviations from the mean

Initial data

y	X: intercept	x1	x2	
3	1	3	5	
1	1	1	4	
8	1	5	6	
3	1	2	4	
5	1	4	6	
4		3	5	mean
5.6		2	0.8	varp
2.366432		1.414214	0.894427	stdp

X'X			X'y
5	15	25	20
15	55	81	76
25	81	129	109

intercept 4

Table 2: Mean-adjusting and standardizing

Mean-adjusted, standardized data = (var-mean)/std

y*		X x1*	x2*	
-0.42258		0	0	
-1.26773		-1.41421	-1.11803	
1.690309		1.414214	1.118034	
-0.42258		-0.70711	-1.11803	
0.422577		0.707107	1.118034	
0		0	0	mean
1		1	1	varp
1		1	1	stdp

X'X		X'y
5	4.743416	4.780914
4.743416	5	4.2521

(intercept	0)			
bx1*	1.494036	=	2.5	= bx1* σy* / σx1*
bx2*	-0.56695	=	-1.5	= bx2* σy* / σx1*
	0.927089		1	

Divided by their std, but retaining means

ys	X x0s	x1s	x2s	
1.267731	1	2.12132	5.59017	
0.422577	1	0.707107	4.472136	
3.380617	1	3.535534	6.708204	
1.267731	1	1.414214	4.472136	
2.112886	1	2.828427	6.708204	
1.690309	1	2.12132	5.59017	mean
1	0	1	1	varp
1	0	1	1	stdp

X'X			X'y
5	10.6066	27.95085	8.451543
10.6066	27.5	64.03612	22.70934
27.95085	64.03612	161.25	51.49766

bx0s	1.690309	4	= bx0s *σy = ysmean * σy
bx1s	1.494036	2.5	= bx1s* σy / σx1
bx2s	-0.56695	-1.5	= bx2s* σy / σx2
	2.617398	5	

Let us observe carefully the atypical regression. Note that a) the values of the ―y‖ in the regressions of Table 1 now appear as those of another ―exogenous variable‖; b) the usual intercept of ones has been moved to the left-hand-side; and c) the regression is run without an intercept. From a financial point to view, whether the vector [3 1 8 3 5] is a security or a portfolio does not matter. What matters is that the optimization will insure that each variable is properly valued and arbitraged in reference to the other securities. In passing, this table also makes obvious that the ―tangent‖ portfolio, in addition to implementing arbitrage conditions, is also an optimal predictor. This means it performs best within the class of linear estimators, under some conditions, and also the best in a larger class of estimators that do not include linearity restrictions. D1, D2, and D3 refer to the determinants of order (k) in the matrix A. Their values (positive for all ranks) indicate that A is positive definite, as any variance-covariance matrix must be.

Table 4 presents the results we are after. A simple standardization, dividing every observation by its standard deviation provides the sought-after standardized mean-correlation system. The correlation matrix can be calculated using matrix functions in EXCEL with the variance covariance-formula when the variables have been standardized: target cells = mmult(transpose(data range – mean vector), data – mean vector).

Note that the values in the right-hand-side vector, [1.690309, 2.12132, 5.59017]', are the values of the means of the original variables divided by their standard deviation: that is, 1.690309 = 4/2.366432. This means the correlation optimization has the correlation table in the left-hand-side and r_i/std_i in the right-hand-side. Noticing this is critical to establish the analytical equivalency of mean-variance and correlation-based optimizations and is something that remains implicit in the numerical examples.

The investor performing the optimizations should have two objectives. The first one is to calculate the optimization numbers, which are found as the solution to the simultaneous equation system A x = b, where A and b are now the correlation matrix and the mean for the standardized data, respectively. The solution vector is bxs = [31.55243 -78.9603
53.66563]', which must be normalized to function as optimal portfolio weights (w_is^*) for the standardized variables. The second objective is to trace these weights back to the original variables, which requires dividing the solution coefficients by the standard deviation of the corresponding original variable (e.g., 13.33333333 = bx_is / $stdx_i$ = 31.55242551 / 2.366431913); this operation returns the non-normalized solutions to the mean-variance optimization (bottom of Table 3, bx and w_i^*). The values we found last must be normalized to provide the original portfolio weights –that is, w_1^* = ($bx_1s/stdx_1$) / sum($bx_is/stdx_i$) =

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13.33333333 / 17.5 = 0.761904762). See appendix for further detail and analytical proof.

Table 3: From regressions to optimal mean-variance portfolios

"Portfolio" data: Obtaining optimal portfolio weights with regressions

ones		x1	x2	x3	
1		3	3	5	
1		1	1	4	
1		8	5	6	
1		3	2	4	
1		5	4	6	
1		4	3	5	mean
0		5.6	2	0.8	varp
0		2.366432	1.414214	0.894427	stdp

X'X				X'y
108	76	109		20
76	55	81		15
109	81	129		25

		wi*
bx1	0.071365	0.761905
bx2	-0.29884	-3.19048
bx3	0.321142	3.428571
	0.093666	1

Conventional mean-variance portfolio optimization

x1	x2	x3	
3	3	5	
1	1	4	
8	5	6	
3	2	4	
5	4	6	
4	3	5	mean
5.6	2	0.8	varp
2.366432	1.414214	0.894427	stdp

variance-covariance matrix				means
5.6	3.2	1.8		4
3.2	2	1.2		3
1.8	1.2	0.8		5

		wi*		
bx1	13.33333	0.761905	D1 =	5.6
bx2	-55.8333	-3.19048	D2 =	0.96
bx3	60	3.428571	D3 =	0.048
Sum bxi	17.5	1		

Table 4: Standardized variables and mean-correlation optimization

Portfolios with regressions: standardized variables

X

ones	x1s	x2s	x3s	
1	1.267731	2.12132034	5.59017	
1	0.422577	0.70710678	4.472136	
1	3.380617	3.53553391	6.708204	
1	1.267731	1.41421356	4.472136	
1	2.112886	2.82842712	6.708204	
1	1.690309	2.12132034	5.59017	mean
0	1	1	1	varp
0	1	1	1	stdp

X'X			X'y
19.28571	22.70934	51.4976594	8.451543
22.70934	27.5	64.0361226	10.6066
51.49766	64.03612	161.25	27.95085

		wis*		wi*	
bx1s	0.16888	0.071365	= bx1s/stdx1	0.761905	= wi*s/sumwi*s
bx2s	-0.42262	-0.29884	= bx1s/stdx2	-3.19048	
bx3s	0.287238	0.321142	= bx1s/stdx3	3.428571	
	0.033494	0.093666		1	

Correlation-table portfolio optimization

X

x1s	x2s	x3s	
1.267731	2.12132034	5.59017	
0.422577	0.70710678	4.472136	
3.380617	3.53553391	6.708204	
1.267731	1.41421356	4.472136	
2.112886	2.82842712	6.708204	
1.690309	2.12132034	5.59017	mean
1	1	1	varp
1	1	1	stdp

correlation matrix			means
1	0.956183	0.85042006	1.690309
0.956183	1	0.9486833	2.12132
0.85042	0.948683	1	5.59017

		wis*	bxis/stdxi	wi*	
bx1s	31.55243	5.042096	13.3333333	0.761905	(1)
bx2s	-78.9603	-12.6179	-55.833333	-3.19048	
bx3s	53.66563	8.575799	60	3.428571	
	6.2578	1	17.5	1	

(1) = (bxis/stdxi)/sum(bxis/stdxi)

4. Concluding Comments

The apparent attractiveness of correlation concepts in portfolio optimization initially motivated this research. By studying the effects of variable transformations in regressions we quickly ascertained how to perform portfolio optimizations using mean-correlation, instead of mean-variance analysis. Of course, the two alternative set-ups produce equivalent optimal portfolio weights if correlation-based number are transformed back to mean-variance ones. Without such transformation, correlation-based optimizations present some a priori advantages –clarification of risk-relations among securities and direct use of return-to-risk measures which link individual return-to-risk and portfolio ratios through the optimal weights. The advantages, however, are outweighed by the inherent limitations –the optimizations effect a transformation that makes interpreting the resulting weights difficult and may deform the risk-structure of the data in a nonlinear and hard-to-assess manner. On a more positive note, the analysis presented strengthens the role of regression methods in portfolio analysis. Further, the difficulty in finding transformations of the data that would both clarify the relationship between individual security characteristics and portfolio weights in a practical manner. In sum, correlation concepts clearly have some pedagogical value, and correlations do provide easy-to-understand information on potential diversification (or lack thereof) that one can obtain before running the optimization.

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